MULTIFRACTAL ANALYSIS OF GENERALISED TAKAGI FUNCTIONS ON THE REAL LINE (Integrated Research on the Theory of Random Dynamical Systems)

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ABSTRACT. We investigate the random iteration of finitely many expanding $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms on the real-line without a common fixed point. We derive the spectral gap property of the associated transition operator on Hölder spaces. As an application we introduce generalised Takagi functions on the real-line and investigate their regularity properties by means of the pointwise Hölder exponents of these functions.

1. INTRODUCTION AND STATEMENT OF RESULTS

For $s \geq 1$, $I := \{1, \dots, s+1\}$ and $\varepsilon > 0$ let $f_i : \mathbb{R} \to \mathbb{R}$, $i \in I$, be $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms with ε-Hölder continuous derivatives. Throughout, we assume that $(f_i)_{i\in I}$ is expanding, that is, there exists $\lambda > 1$ such that $f_i'(x) \geq \lambda > 1$, for all $x \in \mathbb{R}$ and $i \in I$. Further, we assume that $(f_i)_{i\in I}$ has no common fixed point in $\mathbb{R}$. To state our third standing assumption we denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ the two-point compactification of $\mathbb{R}$ and we define $f_i(\pm\infty) := \pm\infty$, $i \in I$. We assume that $(f_i)_{i\in I}$ is contracting near infinity, that is, there exist neighbourhoods $V^{\pm}$ of $\pm\infty$ such that $f_{i|V^{\pm}}$ has Lipschitz norm strictly less than one. Here, $\overline{\mathbb{R}}$ is endowed with a metric d which is strongly equivalent to the Euclidean metric on compact subsets of $\mathbb{R}$.

For $\mathbf{p} = (p_1, \dots, p_s) \in (0,1)^s$ with $\sum_{i=1}^s p_i < 1$, let $p_{s+1} := 1 - \sum_{i=1}^s p_i$. Let $\mu_{\mathbf{p}}$ denote the Bernoulli measure on $I^{\mathbb{N}}$ with probability vector $(p_i)_{i\in I}$. Let $\mathscr{C}(\overline{\mathbb{R}})$ denote the Banach space of continuous function endowed with the supremum norm $\|\cdot\|_\infty$. We define the transition operator for the random walk on $\overline{\mathbb{R}}$ associated with $\mu_{\mathbf{p}}$,

$$M_{\mathbf{p}} : \mathscr{C}\left(\overline{\mathbb{R}}\right) \to \mathscr{C}\left(\overline{\mathbb{R}}\right), \quad M_{\mathbf{p}} h = \int h \circ f_{\omega_1} \, d\mu_{\mathbf{p}}(\omega) = \sum_{i \in I} p_i \cdot h \circ f_i, \; h \in \mathscr{C}\left(\overline{\mathbb{R}}\right).$$

Note that, for each $\alpha > 0$, we have $M_{\mathbf{p}}\left(\mathscr{C}^{\alpha}(\overline{\mathbb{R}})\right) \subset \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$, where $\mathscr{C}^{\alpha}(\overline{\mathbb{R}})$ denotes the Banach space of α-Hölder continuous functions. To state our first main result we say that $M_{\mathbf{p}} : \mathscr{C}^{\alpha}(\overline{\mathbb{R}}) \to \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$ has the *spectral gap property* if its spectrum consists of finitely many eigenvalues of modulus one, and the rest of the spectrum is contained in a ball of radius strictly less than one. For $\mathbf{a} \in \mathbb{R}^s$ and $\delta > 0$ we denote by $B(\mathbf{a}, \delta) \subset \mathbb{R}^s$ the open ball of radius δ with centre $\mathbf{a}$ in $\mathbb{R}^s$.

Theorem 1.1. *For every $\mathbf{p}_0 \in (0,1)^s$ there exist $\delta > 0$ and $\alpha > 0$ such that $M_{\mathbf{p}} : \mathscr{C}^{\alpha}(\overline{\mathbb{R}}) \to \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$ has the spectral gap property for every $\mathbf{p} \in B(\mathbf{p}_0, \delta)$.*

As in [Sum11] we define the probability of tending to infinity

$$(1.1) \qquad T_{\mathbf{p}} : \overline{\mathbb{R}} \to [0,1], \quad T_{\mathbf{p}}(x) := \mu_{\mathbf{p}}\left\{\omega \in I^{\mathbb{N}} \mid \lim_{n\to\infty} f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x) = \infty\right\}.$$

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By combining Theorem 1.1 with perturbation theory for linear operators we can derive that $T_{\mathbf{p}}$ depends real analytically on $\mathbf{p}$. This allows us to make the following definition.

Definition 1.2. We denote by $\mathscr{T} := \mathscr{T}_{\mathbf{p}}$ the $\mathbb{R}$-vector space of generalised Takagi functions generated by

$$C_{\mathbf{n}}(x) := C_{\mathbf{n},\mathbf{p}}(x) := \frac{\partial^{\sum_{i=1}^{s} n_i}}{\partial u_1^{n_1} \partial u_2^{n_2} \dots \partial u_s^{n_s}} T_{(u_1,\dots,u_s)}(x)\big|_{(u_1,\dots,u_s)=\mathbf{p}}, \quad \mathbf{n} = (n_1,\dots,n_s) \in \mathbb{N}_0^s,\ x \in \overline{\mathbb{R}}.$$

We then proceed to investigate the regularity of elements of $\mathscr{T}$. The *pointwise Hölder exponent* of $C \in \mathscr{T}$ at $x \in \mathbb{R}$ is denoted by $\text{Höl}(C,x)$. By [JS15, Lemma 5.1] we have for every $x \in \mathbb{R}$,

$$\text{(1.2)} \qquad \text{Höl}(C,x) = \liminf_{r \to 0} \frac{\log \sup_{y \in B(x,r)} |C(y) - C(x)|}{\log r}.$$

We denote by $G := \langle f_1, \dots f_{s+1} \rangle$ the semigroup generated by $f_1, \dots, f_{s+1}$ where the semigroup operation is the composition of functions. The *Julia set of G* is defined as

$$J := \left\{ x \in \overline{\mathbb{R}} \mid G \text{ is not equicontinuous in any neighborhood } U \text{ of } x \right\}.$$

For the definition of the positive numbers α_- and α_+ we refer to Section 2.3. By t^* we denote the Legendre transform of the function $t = t(\beta)$, $\beta \in \mathbb{R}$, defined implicitly by the pressure formula $\mathscr{P}(t(\beta)\varphi + \beta\psi) = 0$ (see Section 2.3 for the definition). We say that $(f_i)_{i \in I}$ satisfies the open set condition if there exists a non-empty bounded open interval $O \subset \mathbb{R}$ such that $f_i^{-1}(O) \subset O$ for all $i \in I$, and $f_i^{-1}(O) \cap f_j^{-1}(O) = \varnothing$ for all $i, j \in I$ with $i \neq j$. We denote by $\dim_H(A)$ the Hausdorff dimension of a set $A \subset \mathbb{R}$ with respect to the Euclidean metric.

Theorem 1.3. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C \in \mathscr{T} \setminus \{0\}$. Then for every $\alpha \in [\alpha_-, \alpha_+]$ we have*

$$\dim_H \{x \in J \mid \text{Höl}(C,x) = \alpha\} = -t^*(-\alpha),$$

and for $\alpha \notin [\alpha_-, \alpha_+]$ we have

$$\{x \in J \mid \text{Höl}(C,x) = \alpha\} = \varnothing.$$

For the global Hölder continuity of elements of $\mathscr{T}$ we prove the following.

Theorem 1.4. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Then for every $C \in \mathscr{T} \setminus \{0\}$*

$$\alpha_- = \sup\left\{ \alpha \geq 0 \mid C \in \mathscr{C}^{\alpha}(\overline{\mathbb{R}}) \right\}.$$

Moreover, we have $T \in \mathscr{C}^{\alpha_-}\left(\overline{\mathbb{R}}\right)$.

We denote by $\|\cdot\|_{\alpha}$ the α-Hölder norm on $\mathscr{C}^{\alpha}(\overline{\mathbb{R}})$.

Corollary 1.5. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. If $\alpha_- < 1$ then we have for every $\alpha_- < \alpha < 1$,*

$$\sup_{n \geq 1} \|M_{\mathbf{p}}^n\|_{\alpha} = \infty.$$

Regarding the existence of points of non-differentiability of elements of $\mathscr{T}$ we prove the following. Let $\mathbf{e}_k \in \mathbb{N}^s$ denote the k-th unit vector in $\mathbb{N}_0^s$, $1 \leq k \leq s$.

Proposition 1.6. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition.*

(1) If $\alpha_- < 1$ then there exists a dense subset $E \subset J$ of positive Hausdorff dimension such that, for every $C \in \mathscr{T} \setminus \{0\}$ and every $x \in E$, C is not differentiable at x.

(2) If $\alpha_- = 1$ then $C_{\mathbf{e}_k}$ is nowhere differentiable on J, for $1 \le k \le s$. Moreover, if $s = 1$ then C_m is nowhere differentiable on J, for every $m \ge 1$.

Higher order derivatives of the classical Takagi function have been considered in [AK06], where it is shown that the classical Takagi function and the higher order derivatives of the Lebesgue singular function for $p = 1/2$ are nowhere differentiable and α-Hölder continuous for every $\alpha < 1$. These results can be derived from our general theory. In fact, let $s = 1$, $p = 1/2$ and let $f_1(x) = 2x$ and $f_2(x) = 2x - 1$. Then we have $\alpha_- = 1$ and Theorem 1.4 implies $\mathscr{T} \subset \bigcap_{\alpha<1} \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$. Further, by Proposition 1.6 (2) we have that the higher order derivatives of the classical Takagi function C_m, $m \ge 1$, are nowhere differentiable on $J = [0,1]$.

Generalized Takagi functions have also been introduced in [HY84]. In [SS91] it is shown that the Lebesgue singular function depends real analytically on the parameter, and its higher order derivatives are considered. We point out that our theory for the space of functions $\mathscr{T}$ is a far-reaching generalization, where we consider an arbitrary finite number of $\mathscr{C}^{1+\varepsilon}$ diffeomorphisms and arbitrary linear combinations of higher-order partial derivatives of the probability of tending to infinity with $s \ge 1$ probability parameters.

Our results have applications to conjugacies of interval maps. In fact, if $(f_i)_{i\in I}$ satisfies the open set condition, then $T_{\mathbf{p}}$ is the conjugacy map between the expanding dynamical system defined by $(f_i)_{i\in I}$ on J and the piecewise linear map on $[0,1]$ with $(s+1)$ full branches and slopes given by $(1/p_i)_{i\in I}$.

The proofs and detailed statements of the results of this paper will be published elsewhere. In the next section, we briefly outline the methods and ideas used to derive our results.

2. On the proofs of the main results

Let $\Sigma := I^{\mathbb{N}}$. Let $I^* := \bigcup_{n\in\mathbb{N}} I^n$. For $\omega \in I^*$ we denote by $|\omega|$ the unique $n \in \mathbb{N}$ such that $\omega \in I^n$. For $\omega = (\omega_1, \dots, \omega_n) \in I^n$ we let $f_{(\omega_1,\dots,\omega_n)} := f_{\omega_n} \circ \cdots \circ f_{\omega_1}$. Also, for $\omega \in I^{\mathbb{N}}$ and $n \in \mathbb{N}$ we put $\omega_{|n} := (\omega_1, \dots, \omega_n) \in I^n$. We define the coding map $\pi : \Sigma \to \mathbb{R}$ given by

$$\bigcap_{n\in\mathbb{N}} (f_{\omega_{|n}})^{-1}(\overline{\mathbb{R}} \setminus V) = \{\pi(\omega)\}, \quad \omega \in \Sigma, \tag{2.1}$$

where $V := V^+ \cup V^-$ and $V^{\pm}$ are the neighbourhoods of $\pm\infty$ witnessing that $(f_i)_{i\in I}$ is contracting near infinity. Note that, since $(f_i)_{i\in I}$ is expanding, the left hand side of (2.1) is a singleton. Thus, π is well defined. It is easy to see that

$$J = \pi(\Sigma).$$

2.1. **Spectral gap property.** The *kernel Julia set of G* ([Sum11]) is given by

$$\mathrm{J}_{\ker} := \bigcap_{g\in G} g^{-1}(J) \subset J.$$

Our assumptions on $(f_i)_{i\in I}$ imply that $\mathrm{J}_{\ker} = \varnothing$. By using results of [Sum11] we can show that $M_{\mathbf{p}}$ is almost periodic. To derive the spectral gap property, we prove the following key lemma, which is motivated by [Sum13].

Lemma 2.1. *For every* $\mathbf{p}_0 \in (0,1)^s$ *there exist* $\delta > 0$, $\alpha > 0$, $n \in \mathbb{N}$ *and constants* $0 < c < 1$ *and* $C > 0$ *such that, for every* $\mathbf{p} \in B(\mathbf{p}_0, \delta)$ *and for every* $h \in \mathscr{C}^{\alpha}(\overline{\mathbb{R}})$,

$$\left|M_{\mathbf{p}}^n h(x) - M_{\mathbf{p}}^n h(y)\right| \leq \left(c\|h\|_\alpha + C\|h\|_\infty\right) d(x,y)^\alpha, \quad x, y \in \overline{\mathbb{R}}.$$

The spectral gap property then follows from [ITM50]. The analyticity of $\mathbf{p} \mapsto T_{\mathbf{p}}$ is a consequence of the perturbation theory for linear operators (see [Kat76]).

2.2. **Functional equations and matrix cocyle estimates.** The next lemma can be proved exactly as in [JS17, Lemma 4.1].

Lemma 2.2. *For every* $n \in \mathbb{N}_0^s$ *we have*

$$C_{\mathbf{n}} = M_{\mathbf{p}} C_{\mathbf{n}} + \sum_{i=1}^{s} n_i \left(C_{\mathbf{n}-\mathbf{e}_i} \circ f_i - C_{\mathbf{n}-\mathbf{e}_i} \circ f_{s+1}\right).$$

The previous lemma is best stated in terms of a matrix cocycle as in [JS17]. We use $\mathbf{n} = (n_1, \ldots, n_s)$ to denote an element of $\mathbb{N}_0^s$ and we write $|\mathbf{n}| := \sum_{i=1}^s n_i$. Let $\mathbf{e}_i \in \mathbb{N}_0^s$ denote the element whose ith component is 1 and all other components are 0. We denote by $1_{\mathbf{n},\mathbf{m}} \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ the matrix such that for every $(\mathbf{x}, \mathbf{y}) \in \mathbb{N}_0^s \times \mathbb{N}_0^s$ the $(\mathbf{x}, \mathbf{y})$-component of $1_{\mathbf{n},\mathbf{m}}$ is given by

$$(1_{\mathbf{n},\mathbf{m}})_{\mathbf{x},\mathbf{y}} = \begin{cases} 1, & \mathbf{n} = \mathbf{x},\ \mathbf{m} = \mathbf{y} \\ 0, & \text{else.} \end{cases}$$

We define the matrix cocycle $A_0 : \Sigma \times \mathbb{N} \to \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}$ given by

$$A_0(\omega, 1) := \begin{cases} \sum_{\mathbf{n} \in \mathbb{N}_0^s} \left(p_{\omega_1} 1_{\mathbf{n},\mathbf{n}} + n_{\omega_1} 1_{\mathbf{n},\mathbf{n}-\mathbf{e}_{\omega_1}}\right), & \omega_1 \in \{1, \ldots, s\} \\ \sum_{\mathbf{n} \in \mathbb{N}_0^s} \left(p_{\omega_1} 1_{\mathbf{n},\mathbf{n}} - \sum_{i=1}^s n_i 1_{\mathbf{n},\mathbf{n}-\mathbf{e}_i}\right), & \omega_1 = s+1 \end{cases}$$

and we set

$$A_0(\omega, k) := A_0(\omega, 1) A_0(\sigma\omega, 1) \ldots A_0(\sigma^{k-1}\omega, 1) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

We also define

$$A(\omega, k) := (p_{\omega_{|k}})^{-1} A_0(\omega, k) \in \mathbb{R}^{\mathbb{N}_0^s \times \mathbb{N}_0^s}.$$

Moreover, for all $a, b \in \mathbb{R}$ we define the matrix

$$U(a,b) := (u_{\mathbf{n}}(a,b))_{\mathbf{n} \in \mathbb{N}_0^s} \in \mathbb{R}^{\mathbb{N}_0^s} \quad \text{given by } u_{\mathbf{n}}(a,b) := C_{\mathbf{n}}(a) - C_{\mathbf{n}}(b).$$

The purpose of the above definitions is the following.

Lemma 2.3. *Suppose that* $(f_i)_{i\in I}$ *satisfies the open set condition. Let* $k \in \mathbb{N}$, $\omega \in I^k$ *and* $x, y \in f_{\overline{\omega}}^{-1}(\overline{O})$. *Then we have* $U(x,y) = A_0(\overline{\omega}, k) U(f_\omega(x), f_\omega(y))$.

Remark 2.4. The following lemma can be proved as in [JS17, Lemma 4.8].

Lemma 2.5. *There exists a constant $K \geq 1$ which depends only on $\mathbf{p} \in (0,1)^s$ and $\mathbf{q} \in \mathbb{N}^s$ such that for all $\tau \in \Sigma$ and all $k \in \mathbb{N}$,*

$$|A(\tau,k)_{\mathbf{q},\mathbf{r}}| \leq Kk^{|\mathbf{q}|}.$$

Moreover, we can derive the following key lemma from the proof of [JS17, Lemma 5.2]. Note that in [JS17, Lemma 5.2] the Julia set J_ω should be replaced by $J(G)$. An element $C \in \mathscr{T}$ is called non-trivial if there exists $(\beta_{\mathbf{n}})_{\mathbf{n}} \neq 0$ such that $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}}$.

Lemma 2.6. *Suppose that $(f_i)_{i \in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathscr{T}$ be non-trivial. Let $j(k) \to \infty$ be a sequence of positive integers. Let $\omega \in \Sigma$. In any non-empty neighbourhood V in $\mathbb{R}$ which intersects J there exist $a, b \in V \cap O$ with $a \neq b$ such that*

$$\eta := \limsup_{k \to \infty} \left| \sum_{\mathbf{m}} \sum_{\mathbf{n}} \beta_{\mathbf{n}} A(\omega, j(k))_{\mathbf{n},\mathbf{m}} u_{\mathbf{m}}(a,b) \right| \in (0,\infty].$$

2.3. **Multifractal analysis.** To key is to establish a dynamical characterisation of the pointwise Hölder exponent. To this end, we define the potentials

$$\varphi : \Sigma \to \mathbb{R}, \quad \varphi(\omega) := -\log \left| f'_{\omega_1}(\pi(\omega)) \right|, \quad \text{and} \quad \psi := \psi_{\mathbf{p}} : \Sigma \to \mathbb{R}, \quad \psi(\omega) := \log p_{\omega_1}.$$

We define the shift map $\sigma : \Sigma \to \Sigma$, $\sigma((\omega_1, \omega_2, \dots)) := (\omega_2, \omega_3, \dots)$. For $u : \Sigma \to \mathbb{R}$ and $n \in \mathbb{N}$ we denote by $S_n u := \sum_{k=0}^{n-1} u \circ \sigma^k$ the nth ergodic sum. Further we let

$$\alpha_- := \alpha_-(\mathbf{p}) := \inf_{\omega \in \Sigma} \liminf_{n \to \infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)}, \quad \alpha_+ := \alpha_+(\mathbf{p}) := \sup_{\omega \in \Sigma} \limsup_{n \to \infty} \frac{S_n \psi_{\mathbf{p}}(\omega)}{S_n \varphi(\omega)},$$

and we refer to α_- as the bottom of the spectrum. We define

$$\mathscr{F}(\alpha) := \mathscr{F}_{\mathbf{p}}(\alpha) := \pi \left\{ \omega \in \Sigma \mid \lim_{n \to \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} = \alpha \right\}.$$

It is well known that the multifractal spectrum is complete ([Sch99]), that is, we have $\mathscr{F}(\alpha) \neq \varnothing$ if and only if $\alpha \in [\alpha_-, \alpha_+]$. For every $\beta \in \mathbb{R}$ there exists a unique $t(\beta) \in \mathbb{R}$ such that $\mathscr{P}(t(\beta)\varphi + \beta\psi) = 0$, where $\mathscr{P}(u)$ refers to the topological pressure of a continuous function u with respect to the dynamical system (Σ, σ) (see [Wal82]). It is well known that the function t is real-analytic and convex function with $t'(\beta) = -\int \psi \, d\mu_\beta / \int \varphi \, d\mu_\beta$ where μ_β denotes the unique Gibbs probability measure on Σ associated with $t(\beta)\varphi + \beta\psi$. The function t is strictly convex if and only if $\alpha_- < \alpha_+$, and have that $\alpha_- = \alpha_+$ if and only if $\delta\varphi$ and ψ are cohomologous, where

$$\delta := t(0) = \dim_H(J).$$

Here, we say that $\delta\varphi$ and ψ are cohomologous if there exists a continuous function $\kappa : \Sigma \to \mathbb{R}$ such that $\delta\varphi = \psi + \kappa - \kappa \circ \sigma$. Note that we have $-t'(\mathbb{R}) = (\alpha_-, \alpha_+)$ if $\alpha_- < \alpha_+$, and $-t'(\mathbb{R}) = \{\alpha_-\}$, otherwise. We also define the level sets

$$\mathscr{F}^{\#}(\alpha) := \begin{cases} \pi \left\{ \omega \in \Sigma \mid \limsup_{n \to \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} \geq \alpha \right\}, & \alpha \geq \alpha_0 \\ \pi \left\{ \omega \in \Sigma \mid \liminf_{n \to \infty} \frac{S_n \psi(\omega)}{S_n \varphi(\omega)} \leq \alpha \right\}, & \alpha \geq \alpha_0, \end{cases}$$

where we have set $\alpha_0 := \int \psi \, d\mu_0 / \int \varphi \, d\mu_0$. We denote the convex conjugate of t ([Roc70]) by

$$t^*(u) := \sup \{\beta u - t(\beta) \mid \beta \in \mathbb{R}\} \in \mathbb{R} \cup \{+\infty\}.$$

It is well-known (see e.g. [Pes97, Sch99]) that for $\alpha \in [\alpha_-, \alpha_+]$,

$$\dim_H (\mathscr{F}(\alpha)) = \dim_H \left(\mathscr{F}^\#(\alpha)\right) = -t^*(-\alpha) \geq 0, \tag{2.2}$$

and that $\mathscr{F}(\alpha) = \mathscr{F}^\#(\alpha) = \varnothing$ for $\alpha \notin [\alpha_-, \alpha_+]$. To prove this, it is shown that if $-t'(\beta) = \alpha$, for some $\beta \in \mathbb{R}$, then for the corresponding Gibbs measure μ_β we have $\mu_\beta \circ \pi^{-1}(\mathscr{F}(\alpha)) = 1$ and

$$\dim_H (\mathscr{F}(\alpha)) = \dim_H(\mu_\beta \circ \pi^{-1}) > 0. \tag{2.3}$$

We refer to [JS15, JS17] for a closely related framework for random complex dynamical systems. If $\alpha_- = \alpha_+$ then $\mathscr{F}(\alpha_-) = J$ and for every $\beta \in \mathbb{R}$,

$$\dim_H(\mathscr{F}(\alpha_-)) = \dim_H(\mu_\beta \circ \pi_1^{-1}) = \dim_H(J) = t(0) = \delta.$$

By using Lemma 2.6 we are able to prove the following.

Proposition 2.7. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}} \in \mathscr{T}$ be non-trivial. For every $x \in J$ we have*

$$\text{Höl}(C,x) \leq \min_{\omega\in\pi^{-1}(x)} \liminf_{n\to\infty} \frac{S_n\psi(\omega)}{S_n\varphi(\omega)}.$$

The reverse inequality does not hold in general (see Example (3.1) in Section 3). However, we can verify the reverse inequality for almost every $x \in J$ with respect to μ_β by using Lemma 2.3 and Lemma 2.5. Combining this with Proposition 2.7 and the mass distribution principle, we can prove the following lower bound for the Hausdorff dimension in Theorem 1.3.

Proposition 2.8. *Suppose that $(f_i)_{i\in I}$ satisfies the open set condition. Let $C = \sum_{\mathbf{n}} \beta_{\mathbf{n}} C_{\mathbf{n}}$ be non-trivial. Then for all $\alpha \in [\alpha_-, \alpha_+]$ we have*

$$\dim_H \{x \in J \mid \text{Höl}(C,x) = \alpha\} \geq -t^*(-\alpha).$$

Finally, to complete the proof of Theorem 1.3 we extend a result of Allaart ([All17]) for selfsimilar measures to our setting.

Proposition 2.9. *For $\alpha \in \mathbb{R}$ and $C \in \mathscr{T}$ we have* $\dim_H \{x \in J \mid \text{Höl}(C,x) = \alpha\} \leq -t^*(-\alpha)$.

2.4. **Hölder class and non-differentiability.** By using Lemma 2.3 and Lemma 2.5 we are able to verify that

$$\mathscr{T} \subset \bigcap_{\alpha<\alpha_-} \mathscr{C}^\alpha(\mathbb{R}).$$

Moreover, since each level set $\mathscr{F}(\alpha)$, for $\alpha \in [\alpha_-, \alpha_+]$ is non-empty, it follows from Proposition 2.7 that

$$\mathscr{T} \cap \bigcap_{\alpha>\alpha_-} \mathscr{C}^\alpha(\mathbb{R}) = \{0\}.$$

Regarding the non-differentiability of functions in $\mathscr{T}$, it follows from Proposition 2.7 again, that each $C \in \mathscr{T} \setminus \{0\}$ is not differentiability at any point of $\pi\left(\bigcup_{\alpha<1} \mathscr{F}(\alpha)\right)$. Then we can utilize the well-known fact that $\dim_H (\mathscr{F}(\alpha)) > 0$ for every $\alpha \in (\alpha_-, \alpha_+)$.

3. Exceptional points

We provide an example which shows that the inequality in Proposition 2.7 may be strict for systems satisfying the open set condition.

Example 3.1. Let $f_1(x) = 2x$, and $f_2(x) = 2x-1$ and suppose that $p_1 > p_2$. Let $T = T_{p_0}$ denote the probability of tending to infinity. Note that T_p is Lebesgue's singular function. We have

$$\alpha_- = \frac{\log(1/p_1)}{\log 2} = \frac{\psi(\overline{1})}{\varphi(\overline{1})}, \quad \text{and} \quad \alpha_+ = \frac{\log(1/p_2)}{\log 2} = \frac{\psi(\overline{2})}{\varphi(\overline{2})}.$$

There exists a sequence $(n_i)_{i\in\mathbb{N}}$ tending to infinity such that, for $\omega := (12^{n_1}12^{n_2}12^{n_3}\ldots)$ and $x = \pi(\omega)$ we have

$$\text{Höl}(T,x) = \frac{\log(1/p_1)}{\log 2} < \frac{\log(1/p_2)}{\log 2} = \lim_{n\to\infty}\frac{S_n\psi(\omega)}{S_n\varphi(\omega)}.$$

Proof. For $k \in \mathbb{N}$ let

$$\omega^{(k)} := 12^{n_1}12^{n_2}12^{n_3}\ldots 12^{n_k}.$$

Define y_k as the right boundary point of $\pi\left(\left[\omega^{(k-1)}21^{n_k}\right]\right)$, that is

$$y_k := \pi\left(\omega^{(k-1)}21^{n_k}\overline{2}\right).$$

Let

$$r_k := 2\cdot \operatorname{diam}\left(\pi\left[12^{n_1}12^{n_2}12^{n_3}\ldots 12^{n_k}\right]\right) = 2\cdot\exp\left(S_{|12^{n_1}12^{n_2}12^{n_3}\ldots 12^{n_k}|}\varphi(\omega)\right) = 2\cdot\left(\frac{1}{2}\right)^{\sum_{j=1}^{k}(n_k+1)}.$$

We have $y_k \in B(x,r_k)$ because the cylinders $\pi\left[\omega^{(k-1)}12^{n_k}\right]$ and $\pi\left[\omega^{(k-1)}21^{n_k}\right]$ touch, and we have $x\in\pi\left[\omega^{(k-1)}12^{n_k}\right]$ and $y_k\in\left[\omega^{(k-1)}21^{n_k}\right]$. Therefore, we have

$$\begin{aligned}\log \sup_{y\in B(x,r_k)}|T(x)-T(y)| &\geq \log|T(x)-T(y_k)| \geq \log\mu\left[\omega^{(k-1)}21^{n_k}\right]\\ &= S_{|\omega^{(k-1)}21^{n_k}|}\psi(\omega) = S_{|\omega^{(k-1)}|}\psi(\omega)+\log p_1 + n_k\log p_1.\end{aligned}$$

We thus obtain

$$\frac{\log\sup_{y\in B(x,r_k)}|T(x)-T(y)|}{\log r_k} \leq \frac{\log|T(x)-T(y_k)|}{\log r_k} \leq \frac{\log\mu\left[\omega^{(k-1)}21^{n_k}\right]}{\log r_k}.$$

We have

$$\frac{\log\mu\left[\omega^{(k-1)}21^{n_k}\right]}{\log r_k} = \frac{S_{|\omega^{(k-1)}|}\psi(\omega)+\log p_1+n_k\log p_2}{\log 2 + S_{|12^{n_1}12^{n_2}12^{n_3}\ldots 12^{n_k}|}\varphi(\omega)} = \frac{S_{|\omega^{(k-1)}|}\psi(\omega)+\log p_1+n_k\log p_2}{S_{|\omega^{(k-1)}|}\varphi(\omega)+\log 2+(n_k+1)\log(1/2)}$$

Let $\varepsilon > 0$. By choosing n_k much larger than $\left|\omega^{(k-1)}\right|$, we may assume that,

$$\frac{S_{|\omega^{(k-1)}|}\psi(\omega)+\log p_1+n_k\log p_1}{S_{|\omega^{(k-1)}|}\varphi(\omega)+\log 2+(n_k+1)\log(1/2)} \leq \frac{\log p_1}{\log(1/2)}+\varepsilon = \frac{\log(1/p_1)}{\log(2)}+\varepsilon.$$

We have thus shown that

$$\liminf_{r\to 0}\frac{\log\sup_{y\in B(x,r)}|T(x)-T(y)|}{\log r} \leq \liminf_{k\to\infty}\frac{\log\sup_{y\in B(x,r_k)}|T(x)-T(y)|}{\log r_k} \leq \frac{\log(1/p_1)}{\log(2)}+\varepsilon.$$

The claim follows because

$$\mathrm{H\ddot{o}l}(T,x) = \liminf_{r\to 0} \frac{\log \sup_{y\in B(x,r)} |T(x)-T(y)|}{\log r}.$$

□

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